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# Support sizes of threefold quadruple systems[☆]

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ABSTRACT

The support of a quadruple system $QS_\lambda(v)$ is the set of distinct blocks in the design, and the support size is the number of distinct blocks. Let $q_v = \frac{v(v-1)(v-2)}{24}$ and $t_v = q_v + \frac{5v}{3}$. In this paper we show that the set of support sizes of a threefold quadruple system $QS_3(v)$ with $v \equiv 2, 4 \pmod 6$ elements can have any number from $[q_v + 14, 3q_v] \cup \{q_v, q_v + 8, q_v + 12\}$ provided $v \geq 38$, a threefold quadruple system $QS_3(v)$ with $v \equiv 0 \pmod 6$ elements can have any number from $[t_v + 14, 3q_v] \cup \{t_v, t_v + 8, t_v + 12\}$ provided $v \geq 42$.


## 1. Introduction

A *t-wise balanced design* of *order* $v$, block sizes $K$ (assume that each block size in $K$ is not less than $t$) and *index* $\lambda$, or $t-(v, K, \lambda)$ design is a pair $(V, \mathcal{B})$; $V$ is a $v$-set, and $\mathcal{B}$ is a collection of subsets of $V$ called *blocks*, for which each $B \in \mathcal{B}$ has $|B| \in K$, and every $t$-subset of $V$ appears in precisely $\lambda$ of these blocks. When $K = \{k\}$, we only write $k$ for $K$. A $3-(v, 4, \lambda)$ design is called a quadruple system and is denoted by $QS_\lambda(v)$. We will denote a $QS_1(v)$ by $SQS(v)$. A $QS_\lambda(v)$ is called *simple* if there is no repeated block. The existence problem for quadruple systems has been completely settled by Hanani [10,11].

**Theorem 1.1.** *A $QS_\lambda(v)$ exists if and only if $\lambda v \equiv 0 (\mathrm{mod}\ 2)$, $\lambda(v-1)(v-2) \equiv 0 (\mathrm{mod}\ 3)$, $\lambda v(v-1)(v-2) \equiv 0 (\mathrm{mod}\ 8)$ and $v \geq 4$.*

For simple quadruple systems, we have the following result which is devoted by Phelps et al. in [18].

**Theorem 1.2.** *There exists a simple $QS_3(v)$ for any $v \equiv 0 (\mathrm{mod}\ 2)$.*

The main problem we considered in this paper concerns the number of distinct blocks in a $QS_\lambda(v)$. The support of a $QS_\lambda(v)$ is the set of distinct blocks in the design, and the support size is the number of distinct blocks. Foody and Hedayat [8] describe a number of statistical applications for designs with specified support size. For triple systems $TS_\lambda(v)$, that is 2-$(v, 3, \lambda)$ designs, the spectrum of possible support sizes is essentially determined by the work of many researchers [5–7,19,4]. For quadruple systems, however, little is known. Let $QSS(v, \lambda)$ denote the set of support sizes of $QS_\lambda(v)$. Colbourn and Hartman [3] determined many values in $QSS(v, \lambda)$ for $\lambda = 2, 3$. Ajoodani-Namini [2] determined the set $QSS(8m, \lambda)$ for $m \geq 6$ with at most 5 possible omissions for each $m \equiv 0 (\mathrm{mod}\ 3)$.

In this paper, we focus on the threefold quadruple systems. Let $q_v = \frac{v(v-1)(v-2)}{24}$ and $t_v = q_v + \frac{5v}{3}$. From [3,15] we have the following result.

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**Theorem 1.3.** (1) $[q_v + 14,\ 2q_v] \cup \{q_v,\ q_v + 8,\ q_v + 12\} \subset QSS(v, 3)$ *for* $v \equiv 2, 4 (\text{mod } 6)$ *and* $v \geq 16$, $v \neq 26$.
(2) $[t_v + 14,\ 2q_v + \frac{5v}{6}] \cup \{t_v,\ t_v + 8,\ t_v + 12\} \subset QSS(v, 3)$ *for* $v \equiv 0 (\text{mod } 6)$.

We will prove the following theorem in this paper.

**Theorem 1.4.** (1) $(2q_v,\ 3q_v] \subset QSS(v, 3)$ *for* $v \equiv 2, 4$ (mod 6) *and* $v \geq 38$.
(2) $(2q_v + \frac{5v}{6},\ 3q_v] \subset QSS(v, 3)$ *for* $v \equiv 0$ (mod 6) *and* $v \geq 42$.

Let

$$I(v) = \begin{cases} [q_v + 14,\ 3q_v] \cup \{q_v,\ q_v + 8,\ q_v + 12\}, & v \equiv 2, 4 (\text{mod } 6) \\ [t_v + 14,\ 3q_v] \cup \{t_v,\ t_v + 8,\ t_v + 12\}, & v \equiv 0 (\text{mod } 6) \end{cases}$$

we then have the following existence result on QSSs.

**Theorem 1.5.** $I(v) \subset QSS(v, 3)$ *for* $v \equiv 0 (\text{mod } 2)$ *and* $v \geq 38$.

From [2] we know that Theorem 1.5 determines QSS$(v, 3)$ for $v \equiv 2,\ 4 (\text{mod } 6)$. That is we have

**Theorem 1.6.** $I(v) = QSS(v, 3)$ *for* $v \equiv 2,\ 4 (\text{mod } 6)$ *and* $v \geq 38$.

## 2. Intersection of Latin cubes

Let $v$ be a non-negative integer, let $\lambda,\ k$ and $t$ be positive integers. A *group divisible* $t$*-design* (or $t$-GDD) of order $v$ with block size $k$ denoted by GDD$_\lambda$ $(t, k, v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) $X$ is a set of cardinality $v$ (called *points*),
(2) $\mathcal{G} = \{G_1, G_2, \dots\}$ is a set of non-empty subsets of $X$ (called *groups*) such that $(X, \mathcal{G})$ is a 1-design,
(3) $\mathcal{B}$ is a family of subsets of $X$ (called *blocks*) with size $k$, such that each block intersects any given group in at most one point,
(4) each $t$-set of points from $t$ distinct groups is contained in exactly $\lambda$ blocks.

The *type* of $t$-GDD is defined as the multiset $\{|G| : G \in \mathcal{G}\}$. We will denote a GDD$_1$ $(t, k, v)$ by GDD $(t, k, v)$. The *support* of a GDD$_\lambda$ $(t, k, v)$ is the set of distinct blocks in the design, and the *support size* is the number of distinct blocks.

A GDD (3, 4, 24) of type $6^4$ has four groups of size 6. Treating the groups as rows, columns, planes and symbols establishes a one-one correspondence between GDD (3, 4, 24) of type $6^4$ designs and Latin cubes of size 6. Hence we treat the intersection problem for Latin cubes of size 6 here. Three Latin cubes have three-way intersection size $s_1$ if there are precisely $s_1$ cells in which they contain the same symbol, two-way intersection size $s_2$ if there are precisely $s_2$ cells in which two of them contain the same symbol but not in all. We denote by LCI$(n, 3)$ the spectrum of intersection sizes for three Latin cubes of order $n$, that is

$$\text{LCI}(n, 3) = \{(s_1, s_2) : \text{ there are three Latin cubes of order } n, \text{ with three-way intersection size } s_1 \text{ and two-way intersection size } s_2\}.$$

First observe that for three Latin cubes of order 2, the intersection sizes realized are (0, 8) and (8, 0). Now a Latin cube of order 6 can be constructed from 27 Latin cubes of order 2, and hence three Latin cubes of order 6 which intersect in $(8u_1, 8u_2)$ positions exist for $0 \leq u_1,\ u_2 \leq 27$ and $u_1 + u_2 = 27$.

**Lemma 2.1.** $(8u_1, 8u_2) \in$ *LCI(6,3) for* $0 \leq u_1,\ u_2 \leq 27$ *and* $u_1 + u_2 = 27$. □

Similarly, three Latin cubes of order 3 can intersect in (27, 0), (9, 18), (3, 18), (0, 0), (0, 18) and (0, 27) positions. A Latin cube of order 6 can be composed of eight Latin cubes of order 3, and hence three Latin cubes of order 6 which intersect in $(27u_1+9u_2+3u_3,\ 18u_2+18u_3+18u_4+27u_5)$ positions exist for $0 \leq u_1,\ u_2,\ u_3,\ u_4,\ u_5 \leq 8$ and $0 \leq u_1+u_2+u_3+u_4+u_5 \leq 8$.

**Lemma 2.2.** $(27u_1 + 9u_2 + 3u_3, 18u_2 + 18u_3 + 18u_4 + 27u_5) \in LCI(6, 3)$ *for* $0 \leq u_1,\ u_2,\ u_3,\ u_4,\ u_5 \leq 8$ *and* $0 \leq u_1 + u_2 + u_3 + u_4 + u_5 \leq 8$. □

One can also employ three Latin squares of order 6 three-way intersecting $s$ cells, and develop each cyclically into a Latin cube. The resulting Latin cubes three-way intersect in $6s$ positions. Hence for $s \in [0, 19] \cup \{27, 36\}$, $(6s, 0)$ is the intersection size of three Latin cubes of order 6 [1].

**Lemma 2.3.** $(6s, 0) \in LCI(6, 3)$ *for* $s \in [0, 19] \cup \{27, 36\}$. □

We know that two Latin cubes of order 6 can intersect $6s$ positions for $s \in [0, 30] \cup \{32, 36\}$ [9], so three Latin cubes of order 6 can intersect $(6s, 216 - 6s)$ positions for $s \in [0, 30] \cup \{32, 36\}$.

**Lemma 2.4.** $(6s, 216 - 6s) \in LCI(6, 3)$ *for* $s \in [0, 30] \cup \{32, 36\}$. □

We also know that two Latin cubes of order 6 can intersect $s$ positions for $s \in S$ [3], where $S = \{0, 1, 4, 6, 8, 12, 16, 18, 24, 27, 30, 32, 36, 40, 42, 45, 48, 54, 56, 60, 63, 64, 66, 72, 78, 80, 81, 84, 88, 92, 96, 99, 102, 104, 108, 112, 114, 117, 120, 126, 128, 132, 135, 136, 138, 144, 150, 152, 153, 156, 160, 168, 171, 174, 176, 180, 184, 189, 192, 195, 197, 198, 199, 200, 201, 202, 204, 208, 216\}$. So we have

**Lemma 2.5.** $(s, 216 - s) \in LCI(6, 3)$ *for* $s \in S$. □

We also need the following result.

**Lemma 2.6.** $(s_1, s_2) \in LCI(6, 3)$ *for* $(s_1, s_2) \in \{(0, 1), (0, 2), (0, 4), (0, 8)\}$.

**Proof.** We exhibit three cubes intersecting in $(s_1, s_2) \in \{(0, 1), (0, 2), (0, 4), (0, 8)\}$. Here we underline the same value $n$ which is in the same position of two different cubes. □

| $(s_1, s_2) = (0, 1)$ | | | | | |
|---|---|---|---|---|---|
| Cube 1 | | | | | |
| $\underline{0}$12,345 | 234,501 | 450,123 | 123,450 | 345,012 | 501,234 |
| 123,054 | 305,412 | 541,230 | 234,105 | 410,523 | 052,341 |
| 234,501 | 450,123 | 012,345 | 345,012 | 501,234 | 123,450 |
| 305,412 | 541,230 | 123,054 | 410,523 | 052,341 | 234,105 |
| 450,123 | 012,345 | 234,501 | 501,234 | 123,450 | 345,012 |
| 541,230 | 123,054 | 305,412 | 052,341 | 234,105 | 410,523 |
| Cube 2 | | | | | |
| $\underline{0}$43,152 | 315,240 | 524,031 | 401,325 | 132,504 | 250,413 |
| 401,325 | 132,504 | 250,413 | 043,251 | 325,140 | 514,032 |
| 315,240 | 524,031 | 403,152 | 132,504 | 250,412 | 041,325 |
| 132,504 | 250,413 | 041,325 | 325,140 | 514,032 | 403,251 |
| 524,031 | 403,152 | 315,240 | 250,413 | 041,325 | 132,504 |
| 250,413 | 041,325 | 132,504 | 514,032 | 403,251 | 325,140 |
| Cube 3 | | | | | |
| 325,410 | 541,032 | 103,254 | 234,501 | 450,123 | 012,345 |
| 234,501 | 450,123 | 012,345 | 325,410 | 541,032 | 103,254 |
| 541,032 | 103,254 | 325,410 | 450,123 | 012,345 | 234,501 |
| 450,123 | 012,345 | 234,501 | 541,032 | 103,254 | 325,410 |
| 103,254 | 325,410 | 541,032 | 012,345 | 234,501 | 450,123 |
| 012,345 | 234,501 | 450,123 | 103,254 | 325,410 | 541,032 |

| $(s_1, s_2) = (0, 2)$ | | | | | |
|---|---|---|---|---|---|
| Cube 1 | | | | | |
| $\underline{0}$12,345 | 234,501 | 450,123 | 123,450 | 345,012 | 50,1234 |
| 1$\underline{0}$3,254 | 325,410 | 541,032 | 214,305 | 430,521 | 052,143 |
| 234,501 | 450,123 | 012,345 | 345,012 | 501,234 | 123,450 |
| 325,410 | 541,032 | 103,254 | 430,521 | 052,143 | 214,305 |
| 450,123 | 012,345 | 234,501 | 501,234 | 123,450 | 345,012 |
| 541,032 | 103,254 | 325,410 | 052,143 | 214,305 | 430,521 |
| Cube 2 | | | | | |
| $\underline{0}$31,452 | 145,230 | 523,014 | 304,125 | 412,503 | 250,341 |
| 3$\underline{0}$4,125 | 412,503 | 250,341 | 031,452 | 145,230 | 523,014 |
| 145,230 | 523,014 | 301,452 | 412,503 | 250,341 | 034,125 |
| 412,503 | 250,341 | 034,125 | 145,230 | 523,014 | 301,452 |
| 523,014 | 301,452 | 145,230 | 250,341 | 034,125 | 412,503 |
| 250,341 | 034,125 | 412,503 | 523,014 | 301,452 | 145,230 |
| Cube 3 | | | | | |
| 453,120 | 312,045 | 204,531 | 541,302 | 130,254 | 025,413 |
| 541,302 | 130,254 | 025,413 | 453,120 | 312,045 | 204,531 |

$(s_1, s_2) = (0, 2)$

| | | | | | |
|---|---|---|---|---|---|
| 312,045 | 204,531 | 453,120 | 130,254 | 025,413 | 541,302 |
| 130,254 | 035,413 | 541,302 | 312,045 | 204,531 | 453,120 |
| 204,531 | 453,120 | 312,045 | 205,413 | 541,302 | 130,254 |
| 025,413 | 541,302 | 130,254 | 204,531 | 453,120 | 312,045 |

$(s_1, s_2) = (0, 4)$

Cube 1

| | | | | | |
|---|---|---|---|---|---|
| $\underline{0}$12,345 | 234,501 | 450,123 | 123,450 | 345,012 | 501,234 |
| 1$\underline{0}$3,254 | 325,410 | 541,032 | 214,305 | 430,521 | 052,143 |
| 234,5$\underline{0}$1 | 450,123 | 012,345 | 345,012 | 501,234 | 123,450 |
| 325,41$\underline{0}$ | 541,032 | 103,254 | 430,521 | 052,143 | 214,305 |
| 450,123 | 012,345 | 234,501 | 501,234 | 123,450 | 345,012 |
| 541,032 | 103,254 | 325,410 | 052,143 | 214,305 | 430,521 |

Cube 2

| | | | | | |
|---|---|---|---|---|---|
| $\underline{0}$31,452 | 145,230 | 523,014 | 304,125 | 412,503 | 250,341 |
| 3$\underline{0}$4,125 | 412,503 | 250,341 | 031,452 | 145,230 | 523,014 |
| 145,2$\underline{0}$3 | 523,014 | 301,452 | 412,530 | 250,341 | 034,125 |
| 412,53$\underline{0}$ | 250,341 | 034,125 | 145,203 | 523,014 | 301,452 |
| 523,014 | 301,452 | 145,230 | 250,341 | 034,125 | 412,503 |
| 250,341 | 034,125 | 412,503 | 523,014 | 301,452 | 145,230 |

Cube 3

| | | | | | |
|---|---|---|---|---|---|
| 453,120 | 312,045 | 204,531 | 541,302 | 130,254 | 025,413 |
| 541,302 | 130,254 | 025,413 | 453,120 | 312,045 | 204,531 |
| 312,045 | 204,531 | 453,120 | 130,254 | 025,413 | 541,302 |
| 130,254 | 035,413 | 541,302 | 312,045 | 204,531 | 453,120 |
| 204,531 | 453,120 | 312,045 | 205,413 | 541,302 | 130,254 |
| 025,413 | 541,302 | 130,254 | 204,531 | 453,120 | 312,045 |

$(s_1, s_2) = (0, 8)$

Cube 1

| | | | | | |
|---|---|---|---|---|---|
| $\underline{01}$2,345 | 234,501 | 450123 | $\underline{1}$23,45$\underline{0}$ | 345,012 | 501,234 |
| $\underline{10}$3,254 | 325,410 | 541032 | 2$\underline{1}$4,3$\underline{0}$5 | 430,521 | 052,143 |
| 234,501 | 450,123 | 012,345 | 345,012 | 501,234 | 123,450 |
| 325,410 | 541,032 | 103,254 | 430,521 | 052,143 | 214,305 |
| 450,123 | 012,345 | 234,501 | 501,234 | 123,450 | 345,012 |
| 541,032 | 103,254 | 325,410 | 052,143 | 214,305 | 430,521 |

Cube 2

| | | | | | |
|---|---|---|---|---|---|
| $\underline{01}$3,254 | 325,410 | 451,032 | $\underline{1}$04,325 | 432,501 | 251,043 |
| $\underline{10}$2,345 | 234,501 | 450,123 | 0$\underline{1}$3,452 | 345,210 | 521,034 |
| 325,410 | 541,032 | 103,254 | 432,501 | 250,143 | 014,325 |
| 234,501 | 450,123 | 012,345 | 345,210 | 521,034 | 103,452 |
| 541,032 | 103,254 | 325,410 | 250,143 | 014,325 | 432,501 |
| 451,023 | 012,345 | 234,501 | 521,034 | 103,452 | 345,210 |

Cube 3

| | | | | | |
|---|---|---|---|---|---|
| 234,501 | 451,023 | 102,345 | 345,21$\underline{0}$ | 520,134 | 013,452 |
| 325,410 | 540,132 | 013,254 | 432,5$\underline{0}$1 | 251,043 | 104,325 |
| 451,023 | 102,345 | 234,510 | 320,134 | 013,452 | 345,201 |
| 540,132 | 013,254 | 325,401 | 251,043 | 104,325 | 432,510 |
| 102,345 | 234,510 | 451,023 | 013,452 | 345,201 | 520,134 |
| 013,254 | 325,401 | 540,132 | 104,325 | 432,510 | 251,043 |

**Table 2.1**
157 support size of three Latin cubes of order 6.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 216 | 224 | 228 | 230 | 231 | 232 | 233 | 234 | 235 | 237 | 240 | 243 | 246 | 248 | 252 |
| 256 | 258 | 261 | 264 | 270 | 272 | 273 | 276 | 279 | 280 | 282 | 288 | 291 | 294 | 296 |
| 297 | 300 | 303 | 304 | 306 | 309 | 312 | 315 | 318 | 320 | 321 | 324 | 327 | 328 | 330 |
| 333 | 336 | 339 | 340 | 342 | 344 | 345 | 348 | 351 | 352 | 354 | 357 | 360 | 363 | 366 |
| 368 | 369 | 372 | 375 | 376 | 378 | 381 | 384 | 387 | 390 | 392 | 393 | 396 | 399 | 400 |
| 402 | 405 | 408 | 411 | 414 | 416 | 417 | 420 | 423 | 424 | 426 | 428 | 429 | 431 | 432 |
| 435 | 438 | 441 | 444 | 447 | 450 | 453 | 456 | 459 | 462 | 465 | 468 | 471 | 474 | 477 |
| 480 | 483 | 486 | 489 | 492 | 495 | 498 | 501 | 504 | 507 | 510 | 513 | 516 | 519 | 522 |
| 525 | 528 | 531 | 534 | 537 | 540 | 543 | 546 | 549 | 552 | 555 | 558 | 561 | 564 | 567 |
| 570 | 573 | 576 | 579 | 582 | 585 | 588 | 594 | 597 | 600 | 603 | 606 | 612 | 621 | 624 |
| 630 | 636 | 640 | 644 | 646 | 647 | 648 | | | | | | | | |

Combining Lemma 2.1 to Lemma 2.6 we have obtained 659 different intersect sizes in LCI (6, 3). We also have 157 different "support sizes" of three Latin cubes of order 6, where we say the value $s = 648 - (s_1 + s_2)$ is the *support size* of three Latin cubes of order 6. It is easy to see that the support size of three Latin cubes of order 6 is equal to the support size of the corresponding $\mathrm{GDD}_3\,(3, 4, 24)$ of type $6^4$.

Although we have by no means determined all support sizes, we have determined enough for use in the construction. We exploit the fact that many ingredients of $\mathrm{GDD}_3\,(3, 4, 24)$ of type $6^4$ are used in the later constructions of Sections 3, 4. In these constructions, all such ingredients employ the same groups, and any two such ingredients are on at most two common groups. Hence the set of quadruples arising from one $\mathrm{GDD}_3\,(3, 4, 24)$ of type $6^4$ are guaranteed to be disjoint from that of another $\mathrm{GDD}_3\,(3, 4, 24)$ of type $6^4$; we say that the corresponding GDDs are compatible.

**Lemma 2.7.** *For each $k \geq 3$, and each $s$ satisfying $216k \leq s \leq 648k$, $s \neq 216k + t$ for $t \in [1, 13] \setminus \{8, 12\}$, there exist $k$ compatible $\mathrm{GDD}_3$ (3, 4, 24)s of type $6^4$, such that the sum of the support sizes of the GDDs is precisely $s$. That is, there exist $s_1, s_2, \ldots, s_k$ in Table 2.1 such that $s = \sum_{1 \leq i \leq k} s_i$.*

**Proof.** We use mathematical induction to prove the result. For $s$ satisfying $648 \leq s \leq 1944$, $s \neq 648 + t$ for $t \in [1, 13] \setminus \{8, 12\}$, it is easy to check directly that there exist $s_1, s_2, s_3$ in Table 2.1 such that $s = s_1 + s_2 + s_3$. So the result is true for $k = 3$. Assume the result is true for $k = n > 3$. That is, for each $s$ satisfying $216n \leq s \leq 648n$, $s \neq 216n + t$ for $t \in [1, 13] \setminus \{8, 12\}$, there exist $s_1, s_2, \ldots, s_n$ in Table 2.1 such that $s = \sum_{1 \leq i \leq n} s_i$. For $s$ satisfying $216(n+1) \leq s \leq 648(n+1)$, $s \neq 216(n+1)+t$ for $t \in [1, 13] \setminus \{8, 12\}$, there exists a $s_{n+1} \in \{216, 648\}$ such that $s' = s - s_{n+1}$ satisfying $216n \leq s' \leq 648n$, $s' \neq 216n + t$ for $t \in [1, 13] \setminus \{8, 12\}$. We know that the result is true for $k = n + 1$. □

We also need the following result on the support sizes of a $\mathrm{GDD}_3\,(3, 4, 36)$ of type $6^6$.

**Lemma 2.8.** *For $135 \leq s \leq 270$ and $s = 405$, there exists a $\mathrm{GDD}_3$ (3, 4, 36) of type $6^6$ whose support size is $8s$.*

**Proof.** We first consider the case when $135 \leq s \leq 270$. If there exist a GDD (3, 4, 18) of type $3^6$ and a $\mathrm{GDD}_3\,(3, 4, 8)$ of type $2^4$, we can construct a $\mathrm{GDD}_3\,(3, 4, 36)$ of type $6^6$ as follows: develop the assumed GDD (3, 4, 18) by doubling on the sets $A = \{a_1, a_2, \ldots, a_{18}\}$ and $B = \{b_1, b_2, \ldots, b_{18}\}$ so that $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$ is a block if and only if $\{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}$ is so, and put the assumed $\mathrm{GDD}_3\,(3, 4, 8)$ on the set $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\} \cup \{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}$ with the set $\{\{a_{i_j}, b_{i_j}\} : 1 \leq j \leq 4\}$ of groups for every block $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$. Then the family of all such blocks of the $\mathrm{GDD}_3\,(3, 4, 8)$'s on $A \cup B$ is the desired GDD. Since it is known that there are a GDD (3, 4, 18) of type $3^6$ with 135 blocks (cf. [16]) and a $\mathrm{GDD}_3\,(3, 4, 8)$ of type $2^4$ with support size 8 and 16 (from the argument in front of Lemma 2.1), we have a $\mathrm{GDD}_3\,(3, 4, 36)$ of type $6^6$ with support size $8x + 16(135 - x)$ for every $0 \leq x \leq 135$, i.e. $8s$, $135 \leq s \leq 270$.

For the case $s = 405$. If there exist a $\mathrm{GDD}_3\,(3, 4, 18)$ of type $3^6$ and a GDD (3, 4, 8) of type $2^4$, we can construct a $\mathrm{GDD}_3$ (3, 4, 36) of type $6^6$ as follows: develop the assumed $\mathrm{GDD}_3\,(3, 4, 18)$ by doubling on the sets $A = \{a_1, a_2, \ldots, a_{18}\}$ and $B = \{b_1, b_2, \ldots, b_{18}\}$ so that $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$ is a block if and only if $\{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}$ is so, and put the assumed GDD (3, 4, 8) on the set $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\} \cup \{b_{i1}, b_{i2}, b_{i3}, b_{i4}\}$ with the set $\{\{a_{i_j}, b_{i_j}\} : 1 \leq j \leq 4\}$ of groups for every block $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$. Then the family of all such blocks of the GDD (3, 4, 8)'s on $A \cup B$ is the desired GDD. If there is a $\mathrm{GDD}_3\,(3, 4, 18)$ of type $3^6$ with 405 blocks, we then have a $\mathrm{GDD}_3\,(3, 4, 36)$ of type $6^6$ with support size $8 \times 405$ by using the fact that there is a GDD (3, 4, 8) of type $2^4$ with support size 8. We now construct a simple $\mathrm{GDD}_3\,(3, 4, 18)$ of type $3^6$ with 405 blocks as follows:

Point set: $X = Z_3 \times Z_6$.
Group set: $G_i = Z_3 \times \{i\}$, $i \in Z_6$.
Block set:
$\{(i, 0), (i+k, 1), (j, 2), (j+k, 3)\}$, $\{(i, 0), (i+k, 1), (j, 2), (j+k, 4)\}$,
$\{(i, 0), (i+k, 1), (j, 2), (j+k, 5)\}$, $\{(i, 0), (i+k, 1), (j, 3), (j+k, 4)\}$,
$\{(i, 0), (i+k, 1), (j, 3), (j+k, 5)\}$, $\{(i, 0), (i+k, 1), (j, 4), (j+k, 5)\}$,
$\{(i, 2), (i+k, 3), (j, 4), (j+k, 5)\}$, $\{(i, 2), (i+k, 3), (j, 4), (j+k, 0)\}$,
$\{(i, 2), (i+k, 3), (j, 4), (j+k, 1)\}$, $\{(i, 2), (i+k, 3), (j, 5), (j+k, 0)\}$,
$\{(i, 2), (i+k, 3), (j, 5), (j+k, 1)\}$, $\{(i, 4), (i+k, 5), (j, 0), (j+k, 2)\}$,
$\{(i, 4), (i+k, 5), (j, 0), (j+k, 3)\}$, $\{(i, 4), (i+k, 5), (j, 1), (j+k, 2)\}$,
$\{(i, 4), (i+k, 5), (j, 1), (j+k, 3)\}$, $i, j, k \in Z_3$. □

## 3. A construction for QSSs

A *candelabra t-system* (or $t$-CS) of order $v$ is a quadruple $(X, S, \mathcal{G}, \mathcal{B})$ where $X$ is a set of $v$ points, $S$ is a subset (called *stem*) of $X$, $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a partition of $X \setminus S$ into subsets (called *groups* or *branches*) and $\mathcal{B}$ is a family of subsets (called *blocks*) of $X$ which satisfies the following properties:

(1) every $t$-subset $T$ of $X$ with $| T \bigcap (S \bigcup G_i) | < t$ for all $i$ is contained in $\lambda$ blocks,
(2) no $t$-subset of $S \bigcup G_i$ for all $i$ is contained in any block.

A $t$-CS of order $v$ with block sizes from a set $K$ of some positive integers is denoted by $\mathrm{CS}_\lambda\,(t, K, v)$. When $K = \{k\}$, we only write $k$ for $K$. If a $t$-CS has $n_i$ groups of size $g_i$, $1 \le i \le r$, and stem size $s$, then we say that the *group type* (or type) of this system is $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$. A $\mathrm{CS}_\lambda\,(3, 4, v)$ of type $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ is also called a candelabra quadruple system (as in [14]) and is simply denoted by $\mathrm{CQS}_\lambda\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$. We will denote a $\mathrm{CQS}_1\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ by $\mathrm{CQS}\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$.

A $\mathrm{CQS}_\lambda\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ is called *simple* if there is no repeated blocks. The *support* of a $\mathrm{CQS}_\lambda\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ is the set of distinct blocks in the design, and the *support size* is the number of distinct blocks. Let $\mathrm{CQSS}_\lambda\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ denote the set of support sizes of $\mathrm{CQS}_\lambda\,(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$.

For a CQS $(6^u : 0)$, Mills [16] established the following existence result.

**Lemma 3.1.** *There exists a CQS* $(6^u : 0)$ *for any* $u \ge 1$.

**Lemma 3.2.** *There exists a simple* $CQS_3$ $(6^3 : 0)$.

**Proof.** The desired design will be constructed on $Z_6 \times Z_3$ with groups $Z_6 \times \{i\},\ i \in Z_3$. Let

$$\begin{aligned}
&\mathrm{F}_1 = \{\{0,2\},\{3,5\},\{1,4\}\}, && \mathrm{F}_2 = \{\{2,4\},\{1,5\},\{0,3\}\},\\
&\mathrm{F}_3 = \{\{0,4\},\{1,3\},\{2,5\}\}, && \mathrm{F}_4 = \{\{2,3\},\{0,5\},\{1,4\}\},\\
&\mathrm{F}_5 = \{\{1,2\},\{4,5\},\{0,3\}\}, && \mathrm{F}_6 = \{\{0,1\},\{3,4\},\{2,5\}\}.
\end{aligned}$$

The required block set consists of the following four parts:

$$\{\{(x,i),(x+k,i),(y,i+1),(z,i+2)\} : x+y+z = 2i,\ k = 1,2\},$$
$$\{\{(x,i),(x+4,i),(y,i+1),(z,i+2)\} : x+y+z = 2i+1,\ \},$$

where $x, y, z \in Z_6$ and $i \in Z_3$;

$$\{\{(a,i),(b,i),(a',j),(b',j)\} : \{a,b\} \in \mathrm{F}_{2k-1}, \{a',b'\} \in \mathrm{F}_{2k}, k \in \{1,2,3\},\ i,j \in Z_3,\ i \neq j\},$$
$$\{\{(a,i),(b,i),(a',j),(b',j)\} : \{a,b\}, \{a',b'\} \in \mathrm{F}_k,\ k \in \{4,5,6\},\ 0 \le i < j \le 2\}. \quad \square$$

**Lemma 3.3.** *There exists a simple* $CQS_3$ $(6^3 : 2)$.

**Proof.** The desired design will be constructed on $Z_6 \times Z_3 \cup \{\infty_1, \infty_2\}$ with groups $Z_6 \times \{i\},\ i \in Z_3$, and a stem $\{\infty_1, \infty_2\}$. Let

$$\begin{aligned}
&\mathrm{F}_0 = \{\{0,2\},\{3,5\},\{1,4\}\}, && \mathrm{F}_1 = \{\{2,4\},\{1,5\},\{0,3\}\},\\
&\mathrm{F}_2 = \{\{0,4\},\{1,3\},\{2,5\}\}, && \mathrm{F}_3 = \{\{0,1\},\{2,3\},\{4,5\}\},\\
&\mathrm{F}_4 = \{\{1,2\},\{3,4\},\{5,0\}\}, && \mathrm{F}_5 = \{\{0,3\},\{1,4\},\{2,5\}\}.
\end{aligned}$$

The required block set consists of the following six parts:

$$\{\{(x+k,0),(y,1),(z,2),\infty_1\} :\ x+y+z = 0,\ k = 0,2,4\},$$
$$\{\{(x+k,0),(y,1),(z,2),\infty_2\} :\ x+y+z = 0,\ k = 1,3,5\},$$
$$\{\{(x,i),(x+2,i),(y,i+1),(z,i+2)\} :\ x+y+z = 2i,\ i \in Z_3\},$$
$$\{\{(x,i),(x+2,i),(y,i+1),(z,i+2)\} :\ x+y+z = 2i+1,\ i \in Z_3\},$$

where $x, y, z \in Z_6$;

$$\{\{(a,i),(b,i),(a',j),(b',j)\} :\ \{a,b\}, \{a',b'\} \in \mathrm{F}_k,\ k \in \{0,1,2,3,4\},\ 0 \le i < j \le 2\},$$
$$\{\{(a,i),(b,i),(a',j),(b',j)\} :\ \{a,b\} \in \mathrm{F}_k,\ \{a',b'\} \in \mathrm{F}_{k'}, (k,k') = (3,4),(3,5),(4,5),\ i,j \in Z_3,\ i \neq j\}. \quad \square$$

**Lemma 3.4.** *There exists a simple* $CQS_3$ $(6^3 : 4)$.

**Proof.** The desired design will be constructed on $Z_6 \times Z_3 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ with groups $Z_6 \times \{i\},\ i \in Z_3$, and a stem $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. Let

$$F_0 = \{\{0, 2\}, \{3, 5\}, \{1, 4\}\}, \qquad F_1 = \{\{2, 4\}, \{1, 5\}, \{0, 3\}\},$$
$$F_2 = \{\{0, 4\}, \{1, 3\}, \{2, 5\}\}, \qquad F_3 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\},$$
$$F_4 = \{\{1, 2\}, \{3, 4\}, \{5, 0\}\}.$$

The required block set consists of the following seven parts:

$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_1\} : x+y+z = 0,\ k = 0, 2, 4\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_2\} : x+y+z = 0,\ k = 1, 3, 5\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_3\} : x+y+z = 0,\ k = 0, 2, 4\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_4\} : x+y+z = 0,\ k = 1, 3, 5\},$$
$$\{\{(x, i), (x+1, i), (y, i+1), (z, i-1)\} : x+y+z = 2i,\ i \in Z_3\},$$

where $x, y, z \in Z_6$;

$$\{\{(a, i), (b, i), (a', j), (b', j)\} : \{a, b\}, \{a', b'\} \in F_k, k \in \{0, 1, 2, 3, 4\},\ 0 \le i < j \le 2\},$$
$$\{\{(a, i), (b, i), (a', j), (b', j)\} : \{a, b\} \in F_k,\ \{a', b'\} \in F_{k'}, (k, k') = (0, 1), (2, 0), (1, 2), (3, 4),\ i, j \in Z_3,\ i \ne j\}. \quad \square$$

**Lemma 3.5.** *There exists a simple $CQS_3$ $(6^3 : 6)$.*

**Proof.** The desired design will be constructed on $Z_6 \times Z_3 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$ with groups $Z_6 \times \{i\},\ i \in Z_3$, and a stem $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$. Let

$$F_0 = \{\{0, 2\}, \{3, 5\}, \{1, 4\}\}, \qquad F_1 = \{\{2, 4\}, \{1, 5\}, \{0, 3\}\},$$
$$F_2 = \{\{0, 4\}, \{1, 3\}, \{2, 5\}\}, \qquad F_3 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\},$$
$$F_4 = \{\{1, 2\}, \{3, 4\}, \{5, 0\}\}.$$

The required block set consists of the following seven parts:

$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_1\} : x+y+z = 0,\ k = 0, 2, 4\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_2\} : x+y+z = 0,\ k = 1, 3, 5\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_3\} : x+y+z = 0,\ k = 0, 2, 4\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_4\} : x+y+z = 0,\ k = 1, 3, 5\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_5\} : x+y+z = 0,\ k = 0, 2, 4\},$$
$$\{\{(x+k, 0), (y, 1), (z, 2), \infty_6\} : x+y+z = 0,\ k = 1, 3, 5\},$$

where $x, y, z \in Z_6$;

$$\{\{(a, i), (b, i), (a', j), (b', j)\} : \{a, b\}, \{a', b'\} \in F_k, k \in \{0, 1, 2, 3, 4\},\ 0 \le i < j \le 2\},$$
$$\{\{(a, i), (b, i), (a', j), (b', j)\} : \{a, b\} \in F_k,\ \{a', b'\} \in F_{k'}, (k, k') = (0, 1), (2, 3), (4, 0), (1, 2), (3, 4),$$
$$i, j \in Z_3,\ i \ne j\}. \quad \square$$

If a $QS_\lambda(v)$ $(V, \mathcal{B})$ contains a $QS_\lambda(w)$ $(W, \mathcal{B}')$ such that $\mathcal{B}' \subset \mathcal{B}$ and $W \subset V$. We say that the $QS_\lambda(w)$ is a *subdesign* of $(V, \mathcal{B})$. We use the notation $QS_\lambda(v : w)$ to denote a quadruple system of order $v$ missing a subdesign of order $w$. If we allow the subdesign to be missing, we have an incomplete quadruple system which we will also denote by $QS_\lambda(v : w)$. We can regard that the existence of $QS_\lambda(v : w)$ is equivalent to the existence of $QS_\lambda(v)$ for $w = 1, 2$.

**Construction 3.6.** Assume the following exist:

1. a $CQS_\lambda$ $(g^n u^1 : s)$ with support size $r$,
2. $n$ $QS_\lambda$ $(g + s : s)$s, each of which has support size $s_i$, $1 \le i \le n$,
3. a $QS_\lambda$ $(u + s)$ with support size $h$.

Then there exists a $QS_\lambda$ $(ng + u + s)$ with support size $r + h + \sum_{1 \le i \le n} s_i$.

In the following construction, we begin with an $SQS(v)$ $(X \cup \{\infty\}, \mathcal{B})$ in Theorem 1.1 if $v \equiv 2,\ 4 \pmod 6$ or a CQS $(6^{\frac{v}{6}} : 0)$ $(X \cup \{\infty\}, \mathcal{G}, \mathcal{A})$ in Lemma 3.1 if $v \equiv 0 \pmod 6$, and delete point $\infty$ (as in [17]). Let $\mathcal{B}_\infty = \{B \in \mathcal{B} : \infty \in B\}$, $\mathcal{B}' = \mathcal{B} \setminus \mathcal{B}_\infty$. We know that $|\mathcal{B}'| = \frac{(v-1)(v-2)(v-4)}{24}$ and $|\mathcal{B}_\infty| = \frac{(v-1)(v-2)}{6}$. Let $\mathcal{A}_\infty = \{B \in \mathcal{A} : \infty \in B\}$, $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_\infty$. We know that $|\mathcal{A}'| = \frac{(v+3)(v-4)(v-6)}{24}$ and $|\mathcal{A}_\infty| = \frac{(v+3)(v-6)}{6}$. Let $u = \frac{(v-1)(v-2)(v-4)}{24}$, $u_1 = \frac{(v-1)(v-2)}{6}$, $u' = \frac{(v+3)(v-4)(v-6)}{24}$ and $u'_1 = \frac{(v+3)(v-6)}{6}$.

**Construction 3.7.** Let $v \equiv 0 \pmod 2$. Assume the following exist:

1. $u$ $\text{GDD}_\lambda$ $(3, 4, 4g)$s of type $g^4$, each of which has support size $s_i$, $1 \le i \le u$, if $v \equiv 2, 4 \pmod 6$, or $u'$ $\text{GDD}_\lambda$ $(3, 4, 4g)$s of type $g^4$, each of which has support size $s_i$, $1 \le i \le u'$ if $v \equiv 0 \pmod 6$,
2. $\frac{v}{6} - 1$ $\text{GDD}_\lambda$ $(3, 4, 6g)$s of type $g^6$, each of which has support size $h_i$, $1 \le i \le \frac{v}{6} - 1$, if $v \equiv 0 \pmod 6$,
3. a simple $\text{CQS}_\lambda$ $(g^3 : s)$ with $n_1$ blocks,
4. a simple $\text{QS}_\lambda$ $(g + s : s)$ with $n_2$ blocks,
5. a simple $\text{QS}_\lambda$ $(g + s)$ with $m_1$ blocks if $v \equiv 2, 4 \pmod 6$, or a simple $\text{QS}_\lambda$ $(5g + s)$ with $m_2$ blocks if $v \equiv 0 \pmod 6$.

Then there exists a $\text{QS}_\lambda$ $(g(v-1) + s)$ with support size $u_1 n_1 + (v-2)n_2 + m_1 + \sum_{1\le i\le u} s_i$ if $v \equiv 2, 4 \pmod 6$, or with support size $u_1' n_1 + (v-6)n_2 + m_2 + \sum_{1\le i\le u'} s_i + \sum_{1\le i\le \frac{v}{6}-1} h_i$ if $v \equiv 0 \pmod 6$.

**Proof.** We begin with an SQS$(v)$ if $v \equiv 2, 4 \pmod 6$, or a CQS $(6^{\frac{v}{6}} : 0)$ if $v \equiv 0 \pmod 6$, and delete a point to get a 3-$(v-1, \{3, 4\}, 1)$ design $(X, \mathcal{B}' \cup \mathcal{B}'_\infty)$ (where $\mathcal{B}'_\infty = \{B \setminus \{\infty\} : B \in \mathcal{B}_\infty\}$) if $v \equiv 2, 4 \pmod 6$, or a 3-$(v-1, \{3, 4, 5, 6\}, 1)$ design $(X, \mathcal{A}' \cup \mathcal{A}'_\infty \cup \{G : G \in \mathcal{G}, \infty \notin G\} \cup \{G \setminus \{\infty\} : G \in \mathcal{G}, \infty \in G\})$ (where $\mathcal{A}'_\infty = \{B \setminus \{\infty\} : B \in \mathcal{A}_\infty\}$) if $v \equiv 0 \pmod 6$. We shall construct the desired design on $Y = (X \times Z_g) \cup S$, where $S \cap (X \times Z_g) = \emptyset$ and $|S| = s$.

For each block $B_i \in \mathcal{B}'$ or $B_i \in \mathcal{A}'$, construct a $\text{GDD}_\lambda$ $(3, 4, 4g)$ of type $g^4$ with support size $s_i$ on $B_i \times Z_g$ with groups $\{x\} \times Z_g$, $x \in B_i$. Denote its block set by $\mathcal{C}_{B_i}$.

For each block $B_i \in \{G : G \in \mathcal{G}, \infty \notin G\}$, construct a $\text{GDD}_\lambda$ $(3, 4, 6g)$ of type $g^6$ with support size $h_i$ on $B_i \times Z_g$ with groups $\{x\} \times Z_g$, $x \in B_i$. Denote its block set by $\mathcal{C}'_{B_i}$. It is easy to see that the number of blocks in $\{G : G \in \mathcal{G}, \infty \notin G\}$ is $\frac{v}{6} - 1$.

For each block $B \in \mathcal{B}'_\infty$ or $B \in \mathcal{A}'_\infty$, construct a simple $\text{CQS}_\lambda$ $(g^3 : s)$ with $n_1$ blocks on $(B \times Z_g) \cup S$. Denote its block set by $\mathcal{C}''_B$.

If $v \equiv 0 \pmod 6$, for the unique block $B \in \{G \setminus \{\infty\} : G \in \mathcal{G}, \infty \in G\}$, construct a simple $\text{QS}_\lambda$ $(5g + s)$ with $m_2$ blocks on $(B \times Z_g) \cup S$. For each point $x \in X \setminus B$, construct a simple $\text{QS}_\lambda$ $(g + s : s)$ with $n_2$ blocks on $(\{x\} \times Z_g) \cup S$ with a subdesign on $S$. Denote their block set by $\mathcal{C}$.

If $v \equiv 2, 4 \pmod 6$, fix a point $z \in X$, construct a simple $\text{QS}_\lambda$ $(g + s)$ with $m_1$ blocks on $(\{z\} \times Z_g) \cup S$. For each point $x \in X \setminus \{z\}$, construct a simple $\text{QS}_\lambda$ $(g + s : s)$ with $n_2$ blocks on $(\{x\} \times Z_g) \cup S$ with a subdesign on $S$. Denote their block set by $\mathcal{C}'$.

Let $\mathcal{A}^* = (\cup_{1\le i\le u'} \mathcal{C}_{B_i}) \cup (\cup_{1\le i\le \frac{v}{6}-1} \mathcal{C}'_{B_i}) \cup (\cup_{1\le i\le u_1'} \mathcal{C}''_{B_i}) \cup \mathcal{C}$, and
$\mathcal{A}^{**} = (\cup_{1\le i\le u} \mathcal{C}_{B_i}) \cup (\cup_{1\le i\le u_1} \mathcal{C}''_{B_i}) \cup \mathcal{C}'$.

Then $(Y, \mathcal{A}^*)$ is a $\text{QS}_\lambda$ $(g(v-1)+s)$ with support size $u_1' n_1 + (v-6)n_2 + m_2 + \sum_{1\le i\le u'} s_i + \sum_{1\le i\le \frac{v}{6}-1} h_i$ if $v \equiv 0 \pmod 6$, and $(Y, \mathcal{A}^{**})$ is a $\text{QS}_\lambda$ $(g(v-1)+s)$ with support size $u_1 n_1 + (v-2)n_2 + m_1 + \sum_{1\le i\le u} s_i$ if $v \equiv 2, 4 \pmod 6$. □

Combining Lemmas 2.7 and 2.8 we have the following result.

**Lemma 3.8.** *Let $v \equiv 0 \pmod 2$.*

(1) *Assume $v \equiv 2, 4 \pmod 6$ and $\frac{(v-1)(v-2)(v-4)}{24} \ge 3$. If there exist a simple $CQS_3$ $(6^3 : s)$ with $n_1$ blocks, a simple $QS_3$ $(6 + s : s)$ with $n_2$ blocks and a simple $QS_3$ $(6 + s)$ with $m_1$ blocks, then there exists a $QS_3$ $(6(v-1) + s)$ with support size $l$, $w_{6(v-1)+s} \le l \le s_{6(v-1)+s}$, where $w_{6(v-1)+s} = \frac{(v-1)(v-2)}{6} n_1 + (v-2)n_2 + m_1 + 9(v-1)(v-2)(v-4) + 14$ and $s_{6(v-1)+s} = \frac{(v-1)(v-2)}{6} n_1 + (v-2)n_2 + m_1 + 27(v-1)(v-2)(v-4)$.*

(2) *Assume $v \equiv 0 \pmod 6$ and $\frac{(v+3)(v-4)(v-6)}{24} \ge 3$. If there exist a simple $CQS_3$ $(6^3 : s)$ with $n_1$ blocks, a simple $QS_3$ $(6 + s : s)$ with $n_2$ blocks and a simple $QS_3$ $(30 + s)$ with $m_2$ blocks, then there exists a $QS_3$ $(6(v-1) + s)$ with support size $l$, $w'_{6(v-1)+s} \le l \le s'_{6(v-1)+s}$, where $w'_{6(v-1)+s} = \frac{(v+3)(v-6)}{6} n_1 + (v-6)n_2 + m_2 + 9(v-6)(v^2 - v + 8) + 14$ and $s'_{6(v-1)+s} = \frac{(v+3)(v-6)}{6} n_1 + (v-6)n_2 + m_2 + 27(v-6)(v^2 - v + 8)$.*

**Remark.** Moreover, if there also exists a CQS $(6^3 : s)$ with $189 + 27s$ blocks as in [12], then there exists a $\text{QS}_3$ $(6(v-1)+s)$ with more support size $l$, $w_{6(v-1)+s} - (v-1)(v-2)(63 + 9s) \le l \le s_{6(v-1)+s}$ if $v \equiv 2, 4 \pmod 6$, or $w'_{6(v-1)+s} - (v+3)(v-6)(63 + 9s) \le l \le s'_{6(v-1)+s}$ if $v \equiv 0 \pmod 6$.

**Lemma 3.9.** *There exist a simple $QS_3$ $(16 : 4)$ and a simple $QS_3$ $(6 + s : s)$ for $s \in \{4, 6\}$.*

**Proof.** A simple $\text{QS}_3$ $(10 : 4)$ will be constructed on $Z_{10}$ with a subset $\{0, 1, 2, 3\}$. The required 87 blocks are as follows:

{0, 1, 4, 5}, {0, 1, 4, 6}, {0, 1, 4, 7}, {0, 1, 5, 6}, {0, 1, 5, 8}, {0, 1, 6, 9}, {0, 1, 7, 8},
{0, 1, 7, 9}, {0, 1, 8, 9}, {0, 2, 4, 5}, {0, 2, 4, 6}, {0, 2, 4, 7}, {0, 2, 5, 6}, {0, 2, 5, 8},
{0, 2, 6, 9}, {0, 2, 7, 8}, {0, 2, 7, 9}, {0, 2, 8, 9}, {0, 3, 4, 7}, {0, 3, 4, 8}, {0, 3, 4, 9},
{0, 3, 5, 7}, {0, 3, 5, 8}, {0, 3, 5, 9}, {0, 3, 6, 7}, {0, 3, 6, 8}, {0, 3, 6, 9}, {0, 4, 5, 9},
{0, 4, 6, 8}, {0, 4, 8, 9}, {0, 5, 6, 7}, {0, 5, 7, 9}, {0, 6, 7, 8}, {1, 2, 4, 7}, {1, 2, 4, 8},
{1, 2, 4, 9}, {1, 2, 5, 7}, {1, 2, 5, 8}, {1, 2, 5, 9}, {1, 2, 6, 7}, {1, 2, 6, 8}, {1, 2, 6, 9},
{1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 4, 8}, {1, 3, 5, 6}, {1, 3, 5, 9}, {1, 3, 6, 7}, {1, 3, 7, 8},

$\{1,3,7,9\}, \{1,3,8,9\}, \{1,4,5,7\}, \{1,4,6,9\}, \{1,4,8,9\}, \{1,5,6,8\}, \{1,5,7,9\},$
$\{1,6,7,8\}, \{2,3,4,5\}, \{2,3,4,6\}, \{2,3,4,9\}, \{2,3,5,6\}, \{2,3,5,7\}, \{2,3,6,8\},$
$\{2,3,7,8\}, \{2,3,7,9\}, \{2,3,8,9\}, \{2,4,5,9\}, \{2,4,6,8\}, \{2,4,7,8\}, \{2,5,6,7\},$
$\{2,5,8,9\}, \{2,6,7,9\}, \{3,4,5,8\}, \{3,4,6,7\}, \{3,4,7,9\}, \{3,5,6,9\}, \{3,5,7,8\},$
$\{3,6,8,9\}, \{4,5,6,7\}, \{4,5,6,8\}, \{4,5,6,9\}, \{4,5,7,8\}, \{4,6,7,9\}, \{4,7,8,9\},$
$\{5,6,8,9\}, \{5,7,8,9\}, \{6,7,8,9\}.$

For simple $QS_3(16:4)$, it will be constructed on $Z_{16}$ with a subset $\{0,1,2,3\}$. Let

$\mathrm{F}_0 = \{\{0,4\},\{1,6\},\{2,5\},\{3,8\},\{7,10\},\{9,14\},\{11,13\},\{12,15\}\},$
$\mathrm{F}_1 = \{\{0,5\},\{1,7\},\{2,6\},\{3,4\},\{8,10\},\{9,15\},\{11,12\},\{13,14\}\},$
$\mathrm{F}_2 = \{\{0,6\},\{1,5\},\{2,7\},\{3,11\},\{4,13\},\{8,14\},\{9,12\},\{10,15\}\},$
$\mathrm{F}_3 = \{\{0,7\},\{1,4\},\{2,8\},\{3,12\},\{5,11\},\{6,15\},\{9,13\},\{10,14\}\},$
$\mathrm{F}_4 = \{\{0,8\},\{1,10\},\{2,4\},\{3,13\},\{5,14\},\{6,12\},\{7,15\},\{9,11\}\},$
$\mathrm{F}_5 = \{\{0,9\},\{1,11\},\{2,12\},\{3,10\},\{4,15\},\{5,7\},\{6,14\},\{8,13\}\},$
$\mathrm{F}_6 = \{\{0,10\},\{1,15\},\{2,13\},\{3,14\},\{4,9\},\{5,12\},\{6,11\},\{7,8\}\},$
$\mathrm{F}_7 = \{\{0,11\},\{1,12\},\{2,10\},\{3,5\},\{4,14\},\{6,9\},\{7,13\},\{8,15\}\},$
$\mathrm{F}_8 = \{\{0,12\},\{1,9\},\{2,15\},\{3,6\},\{4,10\},\{5,13\},\{7,14\},\{8,11\}\},$
$\mathrm{F}_9 = \{\{0,13\},\{1,14\},\{2,11\},\{3,9\},\{4,8\},\{5,15\},\{6,10\},\{7,12\}\},$
$\mathrm{F}_{10} = \{\{0,14\},\{1,8\},\{2,9\},\{3,15\},\{4,12\},\{5,10\},\{6,13\},\{7,11\}\},$
$\mathrm{F}_{11} = \{\{0,15\},\{1,13\},\{2,14\},\{3,7\},\{4,11\},\{5,6\},\{8,12\},\{9,10\}\},$
$\mathrm{F}_{12} = \{\{4,5\},\{6,7\},\{8,9\},\{10,11\},\{12,13\},\{14,15\}\},$
$\mathrm{F}_{13} = \{\{4,6\},\{5,8\},\{7,9\},\{10,12\},\{11,14\},\{13,15\}\},$
$\mathrm{F}_{14} = \{\{4,7\},\{5,9\},\{6,8\},\{10,13\},\{11,15\},\{12,14\}\}.$

The required block set consists of the following four parts:

$\{\{0,1,a,b\},\ \{2,3,a,b\} : \{a,b\} \in \mathrm{F}_{12}\},$
$\{\{0,2,a,b\},\ \{1,3,a,b\} : \{a,b\} \in \mathrm{F}_{13}\},$
$\{\{0,3,a,b\},\ \{1,2,a,b\} : \{a,b\} \in \mathrm{F}_{14}\},$
$\{\{a,b,c,d\} : \{a,b\},\{c,d\} \in \mathrm{F}_k \text{ and } \{a,b\} \neq \{c,d\},\ k \in Z_{15}\}.$

For simple $QS_3(12:6)$, it will be constructed on $\{0,1,2,3,4,5\} \cup \{0',1',2',3',4',5'\}$ with a subset $\{0',1',2',3',4',5'\}$. Let

$\mathrm{F}_0 = \{\{0,2\},\{1,3\},\{4,5\}\}, \quad \mathrm{F}_1 = \{\{2,4\},\{3,5\},\{0,1\}\},$
$\mathrm{F}_2 = \{\{4,0\},\{5,1\},\{2,3\}\}, \quad \mathrm{F}_3 = \{\{1,2\},\{3,4\},\{5,6\}\},$
$\mathrm{F}_4 = \{\{0,3\},\{1,4\},\{2,5\}\}, \quad \mathrm{F}_5 = \{\{0',2'\},\{1',3'\},\{4',5'\}\},$
$\mathrm{F}_6 = \{\{2',4'\},\{3',5'\},\{0',1'\}\}, \quad \mathrm{F}_7 = \{\{4',0'\},\{5',1'\},\{2',3'\}\},$
$\mathrm{F}_8 = \{\{1',2'\},\{3',4'\},\{5',6'\}\}, \quad \mathrm{F}_9 = \{\{0',3'\},\{1',4'\},\{2',5'\}\}.$

The required block set consists of the following four parts:

$\{\{a,b,a',b'\} : \{a,b\} \in \mathrm{F}_k, \{a',b'\} \in \mathrm{F}_{k+5},\ k \in Z_5\},$
$\{\{a,b,a',b'\} : \{a,b\} \in \mathrm{F}_k, \{a',b'\} \in \mathrm{F}_{k'},\ k+6 \equiv k' \bmod 10,\ k \in Z_5\},$
$\{\{a,b,a',b'\} : \{a,b\} \in \mathrm{F}_k, \{a',b'\} \in \mathrm{F}_{k'},\ k+7 \equiv k' \bmod 10,\ k \in Z_5\},$

and the blocks of a simple $QS_3(6)$ on the set $\{0,1,2,3,4,5\}$. □

## 4. Another construction for QSSs

In order to give another construction for QSSs, we need to introduce *design fragments of types A, B and C* denoted $DFA_3(s)$, $DFB_3(s)$ and $DFC_3(s)$, which are defined as follows: The point set of $DFA_3(s)$ and $DFB_3(s)$ is $(Z_6 \times Z_3) \bigcup \{\infty_1, \ldots, \infty_s\}$, and $DFC_3(s)$ is $Z_6 \times Z_2$. Each consists of blocks of size 4. Blocks in $DFA_3(s)$ and $DFB_3(s)$ are required to intersect each of the three sets $Z_6 \times \{i\},\ i \in Z_3$. Blocks in $DFC_3(s)$ are required to intersect both of the sets $Z_6 \times \{i\},\ i \in Z_2$, in two points. Furthermore, the following conditions are required.

- Triples of the form $\{a_0, b_1, c_2\}$ appear precisely three times in $\mathrm{DFA}_3(s)$ and three times in $\mathrm{DFB}_3(s)$.
- Triples of the form $\{\infty_i, a_0, b_1\}$ appear three times in either one of $\mathrm{DFA}_3(s)$ or $\mathrm{DFB}_3(s)$.
- Triples of the form $\{\infty_i, a_j, b_k\}$ with $j \neq k$ appear precisely three times in $\mathrm{DFA}_3(s)$ ($\mathrm{DFB}_3(s)$) if and only if $\{\infty_i, a_0, b_1\}$ appears in $\mathrm{DFA}_3(s)$ ($\mathrm{DFB}_3(s)$, respectively).
- Triples of the form $\{a_0, b_0, c_1\}$ appear precisely three times in either one of $\mathrm{DFA}_3(s)$, $\mathrm{DFB}_3(s)$ or $\mathrm{DFC}_3(s)$.
- If $\{a_0, b_0, c_1\}$ appears in $\mathrm{DFA}_3(s)$ ($\mathrm{DFB}_3(s)$) then for $i, j \in Z_3,\ i \neq j$, $\{a_i, b_i, c_j\}$ appears in $\mathrm{DFA}_3(s)$ ($\mathrm{DFB}_3(s)$, respectively).
- If $\{a_0, b_0, c_1\}$ appears in $\mathrm{DFC}_3(s)$, then so does the triple $\{a_1, b_1, c_0\}$.

Here we denoted $(i, j) \in Z_6 \times Z_3$ by $i_j$ for simplicity.

Design fragments $\mathrm{DFA}_3(s)$, $\mathrm{DFB}_3(s)$ and $\mathrm{DFC}_3(s)$ are called *simple* if there are no repeated block.

**Lemma 4.1.** *There exist simple $DFA_3(2)$, $DFB_3(2)$ and $DFC_3(2)$.*

**Proof.** Let

$$\mathrm{F}_0 = \{\{5, 0\}, \{1, 4\}, \{2, 3\}\}, \qquad \mathrm{F}_1 = \{\{5, 1\}, \{2, 0\}, \{3, 4\}\},$$
$$\mathrm{F}_2 = \{\{5, 2\}, \{3, 1\}, \{4, 0\}\}, \qquad \mathrm{F}_3 = \{\{5, 3\}, \{4, 2\}, \{0, 1\}\},$$
$$\mathrm{F}_4 = \{\{5, 4\}, \{0, 3\}, \{1, 2\}\}.$$

The block set of $\mathrm{DFA}_3(2)$ consists of the following four parts:

$$\{\{\infty_1, a_0, b_1, c_2\} :\ a + b + c = k,\ k = 0, 2, 4\},$$
$$\{\{\infty_2, a_0, b_1, c_2\} :\ a + b + c = k,\ k = 1, 3, 5\},$$
$$\{\{a_i, (a + 2)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 0), (1, 2), (2, 4)\},$$
$$\{\{a_i, (a + 4)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 1), (1, 3), (2, 5)\},$$

where $a, b, c \in Z_6$.

The block set of $\mathrm{DFB}_3(2)$ consists of the following three parts:

$$\{\{a_i, (a + 1)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 0), (1, 2), (2, 4)\},$$
$$\{\{a_i, (a + 3)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 0), (1, 1), (2, 2)\},$$

where $a, b, c \in Z_6$.

$$\{\{a_i, (a + 3d)_{i+1}, (k - 2a - 3d)_{i+2}, (k + 1 - 2a - 3d)_{i+2}\} :\ k = 1, 3, 5\},$$

where $a \in Z_6, i \in Z_3, b \in \{0, 1\}$.

The block set of $\mathrm{DFC}_3(2)$ is as follows:

$$\{\{a_0, b_0, a'_1, b'_1\} :\ \{a, b\}, \{a', b'\} \in \mathrm{F}_k, k \in \{0, 1, 2, 3, 4\}\}. \quad \square$$

**Lemma 4.2.** *There exist simple $DFA_3(4)$, $DFB_3(4)$ and $DFC_3(4)$.*

**Proof.** Let

$$\mathrm{F}_0 = \{\{0, 2\}, \{3, 5\}, \{1, 4\}\}, \qquad \mathrm{F}_1 = \{\{2, 4\}, \{1, 5\}, \{0, 3\}\},$$
$$\mathrm{F}_2 = \{\{0, 4\}, \{1, 3\}, \{2, 5\}\}, \qquad \mathrm{F}_3 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\},$$
$$\mathrm{F}_4 = \{\{1, 2\}, \{3, 4\}, \{5, 0\}\}.$$

The block set of $\mathrm{DFA}_3(4)$ consists of the following three parts:

$$\{\{\infty_1, a_0, b_1, c_2\}, \{\infty_3, a_0, b_1, c_2\} :\ a + b + c = k,\ k = 0, 2, 4\},$$
$$\{\{\infty_2, a_0, b_1, c_2\}, \{\infty_4, a_0, b_1, c_2\} :\ a + b + c = k,\ k = 1, 3, 5\},$$
$$\{\{a_i, (a + 3)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 0), (1, 1), (2, 2)\},$$

where $a, b, c \in Z_6$.

The block set of $\mathrm{DFB}_3(4)$ consists of the following three parts:

$$\{\{a_i, (a + 1)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 0), (1, 2), (2, 4)\},$$
$$\{\{a_i, (a + 2)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 0), (1, 2), (2, 4)\},$$
$$\{\{a_i, (a + 4)_i, b_{i+1}, c_{i+2}\} :\ a + b + c = j,\ (i, j) = (0, 1), (1, 3), (2, 5)\},$$

where $a, b, c \in Z_6$.

The block set of $\mathrm{DFC}_3(4)$ is as follows:

$$\{\{a_0, b_0, a'_1, b'_1\} :\ \{a, b\}, \{a', b'\} \in \mathrm{F}_k, k \in \{0, 1, 2, 3, 4\}\}.$$
$$\{\{a_0, b_0, a'_1, b'_1\} :\ \{a, b\} \in \mathrm{F}_k,\ \{a', b'\} \in \mathrm{F}_{k'},\ (k, k') = (3, 4), (4, 3)\}. \quad \square$$

In the following construction we also begin with an SQS$(v)$ $(X \cup \{\infty_1, \infty_2\}, \mathcal{B})$ in Theorem 1.1 if $v \equiv 2,\ 4 \pmod 6$ or a CQS $(6^{\frac{v}{6}} : 0)$ $(X \cup \{\infty_1, \infty_2\}, \mathcal{G}, \mathcal{A})$ in Lemma 3.1 if $v \equiv 0 \pmod 6$, but delete two points $\infty_1$ and $\infty_2$ (we need $\infty_1$ and $\infty_2$ to belong the same group $G_0$ in CQS $(6^{\frac{v}{6}} : 0)$ when $v \equiv 0 \pmod 6$). Let $\mathcal{B}_{\{\infty_1,\infty_2\}} = \{B \in \mathcal{B} : \{\infty_1, \infty_2\} \subset B\}$, $\mathcal{B}_{\infty_1} = \{B \in \mathcal{B} \setminus \mathcal{B}_{\{\infty_1,\infty_2\}} : \infty_1 \in B\}$, $\mathcal{B}_{\infty_2} = \{B \in \mathcal{B} \setminus \mathcal{B}_{\{\infty_1,\infty_2\}} : \infty_2 \in B\}$, $\mathcal{B}' = \mathcal{B} \setminus (\mathcal{B}_{\{\infty_1,\infty_2\}} \cup \mathcal{B}_{\infty_1} \cup \mathcal{B}_{\infty_2})$. We know that $|\mathcal{B}'| = \frac{(v-2)(v^2-9v+20)}{24}$, $|\mathcal{B}_{\{\infty_1,\infty_2\}}| = \frac{v-2}{2}$ and $|\mathcal{B}_{\infty_1}| = |\mathcal{B}_{\infty_2}| = \frac{(v-2)(v-4)}{6}$. Let $\mathcal{A}_{\{\infty_1,\infty_2\}} = \{B \in \mathcal{A} : \{\infty_1, \infty_2\} \subset B\}$, $\mathcal{A}_{\infty_1} = \{B \in \mathcal{A} \setminus \mathcal{A}_{\{\infty_1,\infty_2\}} : \infty_1 \in B\}$, $\mathcal{A}_{\infty_2} = \{B \in \mathcal{A} \setminus \mathcal{A}_{\{\infty_1,\infty_2\}} : \infty_2 \in B\}$, $\mathcal{A}' = \mathcal{A} \setminus (\mathcal{A}_{\{\infty_1,\infty_2\}} \cup \mathcal{A}_{\infty_1} \cup \mathcal{A}_{\infty_2})$. We know that $|\mathcal{A}'| = \frac{(v-6)(v^2-5v-12)}{24}$, $|\mathcal{A}_{\{\infty_1,\infty_2\}}| = \frac{v-6}{2}$ and $|\mathcal{A}_{\infty_1}| = |\mathcal{A}_{\infty_2}| = \frac{v(v-6)}{6}$. Let $u = \frac{(v-2)(v^2-9v+20)}{24}$, $u_1 = \frac{(v-2)(v-4)}{6}$, $u' = \frac{(v-6)(v^2-5v-12)}{24}$ and $u'_1 = \frac{v(v-6)}{6}$.

**Construction 4.3.** Let $v \equiv 0 \pmod 2$. Assume the following exist:

1. $u$ GDD$_3$ $(3, 4, 4g)$s of type $g^4$, each of which has support size $s_i$, $1 \le i \le u$, if $v \equiv 2,\ 4 \pmod 6$, or $u'$ GDD$_3$ $(3, 4, 4g)$s of type $g^4$, each of which has support size $s_i$, $1 \le i \le u'$ if $v \equiv 0 \pmod 6$,
2. $\frac{v}{6} - 1$ GDD$_3$ $(3, 4, 6g)$s of type $g^6$, each of which has support size $h_i$, $1 \le i \le \frac{v}{6} - 1$, if $v \equiv 0 \pmod 6$,
3. simple DFA$_3(s)$, DFB$_3(s)$ and DFC$_3(s)$, each of which has $n_1$, $n_2$ and $n_3$ blocks, respectively,
4. a simple QS$_3$ $(12 + s : s)$ with $m_1$ blocks,
5. a simple QS$_3$ $(12 + s)$ with $m_2$ blocks if $v \equiv 2,\ 4 \pmod 6$, or a simple QS$_3$ $(24 + s)$ with $m_3$ blocks if $v \equiv 0 \pmod 6$,

Then there exists a QS$_3$ $(6(v-2)+s)$ with support size $m_2 + \frac{v-4}{2}m_1 + u_1(n_1 + n_2 + 3n_3) + \sum_{1\le i\le u} s_i$ if $v \equiv 2,\ 4 \pmod 6$, or with support size $m_3 + \frac{v-6}{2}m_1 + u'_1(n_1 + n_2) + \frac{(v-6)(v-8)}{2}n_3 + \sum_{1\le i\le u'} s_i + \sum_{1\le i\le \frac{v}{6}-1} h_i$ if $v \equiv 0 \pmod 6$.

**Proof.** We begin with an SQS$(v)$ if $v \equiv 2,\ 4 \pmod 6$ or a CQS $(6^{\frac{v}{6}} : 0)$ if $v \equiv 0 \pmod 6$, and delete two point $\infty_1$ and $\infty_2$ (we need $\infty_1$ and $\infty_2$ to belong the same group $G_0$ when $v \equiv 0 \pmod 6$) to get a CS $(2^{\frac{v-2}{2}} : 0)$ with block sizes $\{3, 4\}$ $(X, \mathcal{B}'_{\{\infty_1,\infty_2\}}, \mathcal{B}' \cup \mathcal{B}'_{\infty_1} \cup \mathcal{B}'_{\infty_2})$ (where $\mathcal{B}'_{\{\infty_1,\infty_2\}} = \{B \setminus \{\infty_1, \infty_2\} : B \in \mathcal{B}_{\{\infty_1,\infty_2\}}\}$, $\mathcal{B}'_{\infty_1} = \{B \setminus \{\infty_1\} : B \in \mathcal{B}_{\infty_1}\}$ and $\mathcal{B}'_{\infty_2} = \{B \setminus \{\infty_2\} : B \in \mathcal{B}_{\infty_2}\}$) if $v \equiv 2,\ 4 \pmod 6$ or a CS $(2^{\frac{v-6}{2}}4^1 : 0)$ with block sizes $\{3, 4, 6\}$ $(X, \mathcal{A}'_{\{\infty_1,\infty_2\}} \cup \{G_0 \setminus \{\infty_1, \infty_2\}\}, \mathcal{A}' \cup \mathcal{A}'_{\infty_1} \cup \mathcal{A}'_{\infty_2} \cup (\mathcal{G} \setminus \{G_0\}))$ (where $\mathcal{A}'_{\{\infty_1,\infty_2\}} = \{B \setminus \{\infty_1, \infty_2\} : B \in \mathcal{A}_{\{\infty_1,\infty_2\}}\}$, $\mathcal{A}'_{\infty_1} = \{B \setminus \{\infty_1\} : B \in \mathcal{A}_{\infty_1}\}$ and $\mathcal{A}'_{\infty_2} = \{B \setminus \{\infty_2\} : B \in \mathcal{A}_{\infty_2}\}$) if $v \equiv 0 \pmod 6$. We shall construct the desired design on $Y = (X \times Z_6) \cup S$, where $S \cap (X \times Z_6) = \emptyset$ and $|S| = s$.

For each block $B_i \in \mathcal{B}'$ or $B_i \in \mathcal{A}'$, construct a GDD$_3$ $(3, 4, 4g)$ of type $g^4$ with support size $s_i$ on $B_i \times Z_g$ with groups $\{x\} \times Z_g$, $x \in B_i$. Denote its block set by $\mathcal{C}_{B_i}$.

For each block $B_i \in \mathcal{G} \setminus \{G_0\}$, construct a GDD$_3$ $(3, 4, 6g)$ of type $g^6$ with support size $h_i$ on $B_i \times Z_g$ with groups $\{x\} \times Z_g$, $x \in B_i$. Denote its block set by $\mathcal{C}'_{B_i}$. It is easy to see that the number of blocks in $\mathcal{G} \setminus \{G_0\}$ is $\frac{v}{6} - 1$.

For each block $B \in \mathcal{B}'_{\infty_1}$ or $B \in \mathcal{A}'_{\infty_1}$, construct a simple DFA$_3(s)$ with $n_1$ blocks on $B \times Z_g$. Denote its block set by $\mathcal{C}''_B$.

For each block $B \in \mathcal{B}'_{\infty_2}$ or $B \in \mathcal{A}'_{\infty_2}$, construct a simple DFB$_3(s)$ with $n_2$ blocks on $(B \times Z_g) \cup S$. Denote its block set by $\mathcal{C}'''_B$.

For each pair $\{x, y\} \subset X$, $\{x, y\} \notin \mathcal{B}'_{\{\infty_1,\infty_2\}}$, or $\{x, y\} \subset X \setminus G_0$, $\{x, y\} \notin \mathcal{A}'_{\{\infty_1,\infty_2\}}$, construct a simple DFC$_3(s)$ with $n_3$ blocks on $\{x, y\} \times Z_g$. Denote its block set by $\mathcal{C}_{\{x,y\}}$. It is easy to see that the number of these pairs is $\frac{(v-2)(v-4)}{2}$ or $\frac{(v-6)(v-8)}{2}$.

Let $\mathcal{A}^* = (\cup_{1\le i\le u'} \mathcal{C}_{B_i}) \cup (\cup_{1\le i\le \frac{v}{6}-1} \mathcal{C}'_{B_i}) \cup (\cup_{1\le i\le u'_1} (\mathcal{C}''_{B_i} \cup \mathcal{C}'''_{B_i})) \cup (\cup_{\{x,y\}\subset X\setminus G_0 \text{ and } \{x,y\}\notin \mathcal{A}'_{\{\infty_1,\infty_2\}}} \mathcal{C}_{\{x,y\}})$, and
$\mathcal{A}^{**} = (\cup_{1\le i\le u} \mathcal{C}_{B_i}) \cup (\cup_{1\le i\le u_1} (\mathcal{C}''_{B_i} \cup \mathcal{C}'''_{B_i})) \cup (\cup_{\{x,y\}\subset X \text{ and } \{x,y\}\notin \mathcal{B}'_{\{\infty_1,\infty_2\}}} \mathcal{C}_{\{x,y\}})$.

Then $(Y, S, \{B \times Z_6 : B \in \mathcal{A}'_{\{\infty_1,\infty_2\}}\} \cup \{(G_0 \setminus \{\infty_1, \infty_2\}) \times Z_6\}, \mathcal{A}^*)$ is a CQS$_3$ $(12^{\frac{v-6}{2}}24 : s)$ with support size $u'_1(n_1 + n_2) + \frac{(v-6)(v-8)}{2}n_3 + \sum_{1\le i\le u'} s_i + \sum_{1\le i\le \frac{v}{6}-1} h_i$ if $v \equiv 0 \pmod 6$, and $(Y, S, \{B \times Z_6 : B \in \mathcal{A}'_{\{\infty_1,\infty_2\}}\}, \mathcal{A}^*)$ is a CQS$_3$ $(12^{\frac{v-2}{2}} : s)$ with support size $u_1(n_1 + n_2) + \frac{(v-2)(v-4)}{2}n_3 + \sum_{1\le i\le u} s_i$ if $v \equiv 2,\ 4 \pmod 6$.

The result then follows from Construction 3.6. $\square$

Combining Lemmas 2.7 and 2.8 we have the following result.

**Lemma 4.4.** *Let $v \equiv 0 \pmod 2$.*

(1) *Assume $v \equiv 2,\ 4 \pmod 6$ and $\frac{(v-2)(v^2-9v+20)}{24} \ge 3$. If there exist a simple QS$_3$ $(12+s : s)$ with $m_1$ blocks, a simple QS$_3$ $(12+s)$ with $m_2$ blocks, and a simple DFA$_3(s)$, DFB$_3(s)$ and DFC$_3(s)$, each of which has $n_1$, $n_2$ and $n_3$ blocks, respectively, then there exists a QS$_3$ $(6(v-2)+s)$ with support size $l$, $d_{6(v-2)+s} \le l \le g_{6(v-2)+s}$, where $d_{6(v-2)+s} = m_2 + \frac{v-4}{2}m_1 + \frac{(v-2)(v-4)}{6}(n_1 + n_2 + 3n_3) + 9(v-2)(v^2-9v+20) + 14$ and $g_{6(v-2)+s} = m_2 + \frac{v-4}{2}m_1 + \frac{(v-2)(v-4)}{6}(n_1 + n_2 + 3n_3) + 27(v-2)(v^2-9v+20)$.*

(2) *Assume $v \equiv 0 \pmod 6$ and $\frac{(v-6)(v^2-5v+12)}{24} \ge 3$. If there exist a simple QS$_3$ $(12+s : s)$ with $m_1$ blocks, a simple QS$_3$ $(24+s)$ with $m_3$ blocks, and simple DFA$_3(s)$, DFB$_3(s)$ and DFC$_3(s)$, each of which has $n_1$, $n_2$ and $n_3$ blocks, respectively, then there exists a QS$_3$ $(6(v-2)+s)$ with support size $l$, $d'_{6(v-2)+s} \le l \le g'_{6(v-2)+s}$, where $d'_{6(v-2)+s} = m_3 + \frac{v-6}{2}m_1 + u'_1(n_1 + n_2) + \frac{(v-6)(v-8)}{2}n_3 + 9(v-6)(v^2-5v+8) + 14$ and $g'_{6(v-2)+s} = m_3 + \frac{v-6}{2}m_1 + u'_1(n_1 + n_2) + \frac{(v-6)(v-8)}{2}n_3 + 27(v-6)(v^2-5v+8)$.*

**Table 5.1**

| Ingredients | Number of blocks of size four |
|---|---|
| Simple$DFA_3(2)$,$DFB_3(2)$,$DFC_3(2)$ | 432, 324, 45 |
| Simple$DFA_3(4)$,$DFB_3(4)$,$DFC_3(4)$ | 540, 324, 63 |
| Simple $CQS_3(6^3 : 0)$ | 567 |
| Simple $CQS_3(6^3 : 2)$ | 729 |
| Simple $CQS_3(6^3 : 4)$ | 891 |
| Simple $CQS_3(6^3 : 6)$ | 1053 |
| Simple $QS_3(10 : 4)$ | 87 |
| Simple $QS_3(12 : 6)$ | 150 |
| Simple $QS_3(16 : 4)$ | 417 |
| Simple $QS_3(6)$ | 15 |
| Simple $QS_3(8)$ | 42 |
| Simple $QS_3(10)$ | 90 |
| Simple $QS_3(12)$ | 165 |
| Simple $QS_3(14)$ | 273 |
| Simple $QS_3(16)$ | 420 |
| Simple $QS_3(26)$ | 1950 |
| Simple $QS_3(28)$ | 2457 |
| Simple $QS_3(30)$ | 3045 |
| Simple $QS_3(32)$ | 3720 |
| Simple $QS_3(34)$ | 4488 |
| Simple $QS_3(36)$ | 5355 |

**Remark.** Moreover, if there also exist DFA($s$) and DFB($s$) with $243 + 27s$ blocks as in [13], then there exists a $QS_3$ $(6(v-2)+s)$ with greater support size $l$, $d_{6(v-2)+s} - (v-2)(v-4)(81+9s) \leq l \leq g_{6(v-2)+s}$ if $v \equiv 2, 4 \pmod 6$, or $d'_{6(v-2)+s} - v(v-6)(81+9s) \leq l \leq g'_{6(v-2)+s}$ if $v \equiv 0 \pmod 6$.

## 5. Proof of Theorem 1.4

In this section we will prove Theorem 1.4. Having determined certain possible support sizes, we now consider in more detail the number of blocks of size 4 in each ingredient used in Lemmas 3.8 and 4.4, however they are tabulated in Table 5.1.

**Proof of Theorem 1.4.** The proof consists of the following two parts.

For the case $v \equiv 2, 4 \pmod{12}$ and $v \geq 38$, write $v = 6u + s$, $s = 2, 4$, where $u \equiv 0 \pmod 2$. We know that there exist an SQS$(u+2)$ for $u \equiv 0, 2 \pmod 6$, and a CQS $(6^{\frac{u+2}{6}} : 0)$ for $u \equiv 4 \pmod 6$. The result for $v \geq 74$ follows from Lemma 4.4 with the facts that $d_{6u+s} < 2q_{6u+s}$ and $g_{6u+s} = 3q_{6u+s}$ for $u \equiv 0, 2 \pmod 6$ and $d'_{6u+s} < 2q_{6u+s}$ and $g'_{6u+s} = 3q_{6u+s}$ for $u \equiv 4 \pmod 6$. Here we need simple $QS_3$ $(12+s : s)$, simple $QS_3$ $(12+s)$, simple $QS_3$ $(24+s)$ and simple $DFA_3(s)$, $DFB_3(s)$ and $DFC_3(s)$ for $s = 2, 4$ as input designs, which come from Theorem 1.2, Lemma 3.9 and Lemmas 4.1 and 4.2. The result for $38 \leq v < 74$ follows from Lemma 4.4 and its Remark with the facts that $d_{6u+s} - (v-2)(v-4)(27+3s) < 2q_{6u+s}$ and $g_{6u+s} = 3q_{6u+s}$ for $u \equiv 0, 2 \pmod 6$ and $d'_{6u+s} - v(v-6)(27+3s) < 2q_{6u+s}$ and $g'_{6u+s} = 3q_{6u+s}$ for $u \equiv 4 \pmod 6$. Here we need DFA($s$) and DFB($s$) for $s = 2, 4$ as input designs, which come from [10,13].

For the case $v \equiv 6, 8, 10, 12 \pmod{12}$ with $v \geq 42$, write $v = 6u + s$, $s = 0, 2, 4, 6$. We know that there exist an SQS$(u+1)$ for $u \equiv 1, 3 \pmod 6$, and a CQS $(6^{\frac{u+1}{6}} : 0)$ for $u \equiv 5 \pmod 6$. The result for $v \geq 42$ and $v \notin \{44, 46, 48, 60\}$ follows from Lemma 3.8 with the facts that $w_{6u+s} < 2q_{6u+s}$ and $s_{6u+s} = 3q_{6u+s}$ for $u \equiv 1, 3 \pmod 6$ and $s \neq 6$, $w_{6u+s} < 2q_{6u+s} + 5u + \frac{5s}{6}$ and $s_{6u+s} = 3q_{6u+s}$ for $u \equiv 1, 3 \pmod 6$ and $s = 6$, $w'_{6u+s} < 2q_{6u+s}$ and $s'_{6u+s} = 3q_{6u+s}$ for $u \equiv 5 \pmod 6$ and $s \neq 6$ and $w'_{6u+s} < 2q_{6u+s} + 5u + \frac{5s}{6}$, and $s'_{6u+s} = 3q_{6u+s}$ for $u \equiv 5 \pmod 6$ and $s = 6$. Here we need simple $CQS_3$ $(6^3 : s)$, simple $QS_3$ $(6+s : s)$, simple $QS_3$ $(6+s)$ and simple $QS_3$ $(30+s)$ for $s = 0, 2, 4, 6$ as input designs, which come from Theorem 1.2, Lemma 3.9 and Lemmas 3.2–3.5. The result for $v \in \{44, 46, 48, 60\}$ follows from Lemma 3.8 and its Remark with the facts that $w_{6u+s} - u(u-1)(63+9s) < 2q_{6u+s}$ and $s_{6u+s} = 3q_{6u+s}$ for $u \equiv 1, 3 \pmod 6$ and $s \neq 6$, $w_{6u+s} - u(u-1)(63+9s) < 2q_{6u+s} + 5u + \frac{5s}{6}$ and $s_{6u+s} = 3q_{6u+s}$ for $u \equiv 1, 3 \pmod 6$ and $s = 6$, $w'_{6u+s} - (u+4)(u-5)(63+9s) < 2q_{6u+s}$ and $s'_{6u+s} = 3q_{6u+s}$ for $u \equiv 5 \pmod 6$ and $s \neq 6$ and $w'_{6u+s} - (u+4)(u-5)(63+9s) < 2q_{6u+s} + 5u + \frac{5s}{6}$, and $s'_{6u+s} = 3q_{6u+s}$ for $u \equiv 5 \pmod 6$ and $s = 6$. Here we need CQS $(6^3 : s)$ for $s = 2, 4, 6$ as input designs, which come from [12]. □

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